Solutions 10

Exercise 2.2.12

Part 1 The conditional Bayes Risk is

$$R(\delta|H_0) = C_{10}P_F(\delta) + C_{00}(1 - P_F(\delta)) \tag{1}$$

and

$$R(\delta|H_1) = C_{11}P_D(\delta) + C_{01}(1 - P_D(\delta)) \tag{2}$$

Hence the Bayes Risk is

$$\begin{aligned} R(\delta, \phi(\mathbf{R})) &= [C_{10}P_F(\delta) + C_{00}(1 - P_F(\delta))](1 - \phi(\mathbf{R})) \\ &\quad + [C_{11}P_D(\delta) + C_{01}(1 - P_D(\delta))]\phi(\mathbf{R}) \\ &= C_{00}(1 - \phi(\mathbf{R})) + C_{01}\phi(\mathbf{R}) \\ &\quad + (C_{10} - C_{00})P_F(\delta)(1 - \phi(\mathbf{R})) + (C_{11} - C_{01})P_D(\delta)\phi(\mathbf{R}), \end{aligned} \tag{3}$$

where $P_F(\delta) = \sum_{y\in\mathcal{Y}_1} P(y|H_0)$ and $P_D(\delta) = \sum_{y\in\mathcal{Y}_1} P(y|H_1)$.

Part 2 The LRT that minimize the Bayes Risk is $L(\delta, \phi(\mathbf{R})) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$, where

$$\tau = \frac{(C_{10} - C_{00})(1 - \phi(\mathbf{R}))}{(C_{11} - C_{01})\phi(\mathbf{R})}. \tag{4}$$

It is a specific decision rule, and the case that $(\delta, \phi(\mathbf{R})) = \tau$ doesn't influence the Bayes Risk since $(C_{10} - C_{00})P_F(\delta)(1 - \phi(\mathbf{R})) + (C_{11} - C_{01})P_D(\delta)\phi(\mathbf{R}) = 0$. Hence a randomized test is not necessary.

Part 3 The slope of and straight-line segment on the ROC is a constant, which is exactly the LRT. For the same LRT, we have the same decision rule hence $P_F(\delta)$ and $P_D(\delta)$ are constant. Then the Bayes Risk is constant.

Part 4 If P_F is continuous function with respect to τ, then $P_F = \alpha$. If it is a discrete function, then we need to find p that

$$p(1 - \alpha^+) + (1 - p)(1 - \alpha^-) = 1 - \alpha \tag{5}$$

and hence $p = \frac{\alpha^- - \alpha}{\alpha^- - \alpha^+}$. Then $\phi(\mathbf{R}) = \frac{\alpha^- - \alpha}{\alpha^- - \alpha^+}$.

Exercise 2.2.15

Part 1

$$\begin{aligned}\sqrt{2\pi}\mathrm{erfc}(x) &= \int_x^{\infty} e^{-\frac{t^2}{2}}\,dt \\ &= \frac{1}{x}e^{-\frac{x^2}{2}} - \int_x^{\infty} \frac{1}{t^2}e^{-\frac{t^2}{2}}\,dt \\ &= f_1(x) \\ &= \frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} + \int_x^{\infty} \frac{3}{t^4}e^{-\frac{t^2}{2}}\,dt \\ &= f_2(x)\end{aligned} \tag{6}$$

We know

$$f_1(x) \le \frac{1}{x}e^{-\frac{x^2}{2}} \tag{7}$$

and

$$\begin{aligned}f_2(x) &\ge \frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} \\ &= \frac{1}{x}e^{-\frac{x^2}{2}}\left(1 - \frac{1}{x^2}\right)\end{aligned} \tag{8}$$

Hence we finish the proof.

Part 2

Solution 1 By mathematical induction we can finish the proof.

Suppose

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}\left[1 + \sum_{m=1}^{n-1}(-1)^m \frac{1\cdot 3\cdots(2m-1)}{x^{2m}} + R_n\right] \tag{9}$$

and

$$R_n = (-1)^n x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1\cdot 3\cdots(2n-1)}{t^{2n}} e^{-\frac{t^2}{2}}\,dt \tag{10}$$

Firstly, from part 1 we have

$$\begin{aligned}\mathrm{erfc}(x) &= \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{t^2}{2}}\,dt \\ &= \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left[1 - xe^{\frac{x^2}{2}}\int_x^{+\infty}\frac{1}{t^2}e^{-\frac{t^2}{2}}\,dt\right] \\ &= \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left[1 + R_1\right] \\ &= \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left[1 - \frac{1}{x^2} + xe^{\frac{x^2}{2}}\int_x^{\infty}\frac{3}{t^4}e^{-\frac{t^2}{2}}\,dt\right] \\ &= \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left[1 - \frac{1}{x^2} + R_2\right]\end{aligned} \tag{11}$$

Then the assumption holds for $n = 1$ and $n = 2$. Suppose that it holds for the k-th item, then for $k+1$-th term we have

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left(1 + \sum_{m=1}^{k-1} (-1)^m \frac{1 \cdot 3 \cdots (2m-1)}{x^{2m}} + R_k \right) \tag{12}$$

where

$$\begin{aligned} R_k &= (-1)^k x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)}{t^{2k}} e^{-\frac{t^2}{2}} dt \\ &= (-1)^k x e^{\frac{x^2}{2}} \left[\frac{1 \cdot 3 \cdots (2k-1)}{x^{2k+1}} e^{-\frac{x^2}{2}} - \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)(2k+1)}{t^{2k+2}} e^{-\frac{t^2}{2}} dt \right] \\ &= (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}} + (-1)^{k+1} x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)(2k+1)}{t^{2k+2}} e^{-\frac{t^2}{2}} dt \\ &= (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}} + R_{k+1} \end{aligned} \tag{13}$$

Hence

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left(1 + \sum_{m=1}^{k} (-1)^m \frac{1 \cdot 3 \cdots (2m-1)}{x^{2m}} + R_{k+1} \right) \tag{14}$$

and the assumption is right.

From (13) we can get R_k has the same sign with $(-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}}$ when $x > 0$, i.e. R_k has the same sign as the $k+1$-th term. Notice that R_{k+1} has the different sign from R_k, then $|R_k| < \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}}$, i.e. the remainder is less than the magnitude of the $k+1$-th term.

Solution 2

$$\begin{aligned} \mathsf{erfc}(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}} - \int_x^{\infty} e^{-\frac{u^2}{2}} \frac{1}{u^2} du \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}} - \frac{1}{x^3} e^{-\frac{x^2}{2}} + \int_x^{\infty} \frac{3}{u^4} e^{-\frac{u^2}{2}} du \\ &\vdots \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}} \left[1 + \sum_{m=1}^{n-1} (-1)^m \frac{1 \cdot 3 \cdot \cdots \cdot (2m-1)}{x^{2m}} \right] \\ &\quad + \frac{1}{\sqrt{2\pi}} (-1)^n 1 \cdot 3 \cdot \cdots \cdot (2n-1) \int_x^{\infty} \frac{1}{u^{2n}} e^{-\frac{u^2}{2}} du \end{aligned} \tag{15}$$

If we want to derive R_n, we need prove that

$$\int_x^{\infty} \frac{1}{u^{2n}} e^{-\frac{u^2}{2}} du = \frac{1}{x} e^{-\frac{x^2}{2}} \frac{1}{x^{2n}} \theta \tag{16}$$

And the proof is:

$$\begin{aligned}\int_x^\infty \frac{1}{u^{2n}} e^{-\frac{u^2}{2}} du &\overset{t=\frac{u^2-x^2}{2}}{\Longrightarrow} \int_0^\infty \frac{1}{(2t+x^2)^n} e^{-(t+\frac{x^2}{2})} \frac{1}{\sqrt{2t+x^2}} dt \\ &= e^{-\frac{x^2}{2}} \cdot \frac{1}{x^{2n+1}} \int_0^\infty e^{-t}(1+\frac{1}{x^2})^{-n-\frac{1}{2}} dt \\ &= e^{-\frac{x^2}{2}} \cdot \frac{1}{x^{2n+1}} \theta\end{aligned} \tag{17}$$

Therefore,

$$\mathsf{erfc}(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left[1 + \sum_{m=1}^{n-1} (-1)^m \frac{1\cdot 3 \cdot \cdots \cdot (2m-1)}{x^{2m}} + R_n\right] \tag{18}$$

where

$$R_n = \left[(-1)^n \cdot \frac{1\cdot 3 \cdot \cdots \cdot (2n-1)}{2^{2n}}\right] \int_0^\infty e^{-t}(1+\frac{2t}{x^2})^{-n-\frac{1}{2}} dt \tag{19}$$

Exercise 2.2.17

Part 1 The LRT is

$$\begin{aligned}L(X_1, X_2) &= \frac{p_{x_1,x_2|H_1}(x_1, x_2|H_1)}{p_{x_1,x_2|H_0}(x_1, x_2|H_0)} \\ &= \frac{\sigma_0 \left[\exp\left(\frac{X_1^2}{2\sigma_0^2} - \frac{X_1^2}{2\sigma_1^2}\right) + \exp\left(\frac{X_2^2}{2\sigma_0^2} - \frac{X_2^2}{2\sigma_1^2}\right)\right]}{2\sigma_1}\end{aligned} \tag{20}$$

Part 2 We have

$$\begin{aligned}P_D &= P(H_1|H_1) \\ &= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1, x_2|H_1) dx_1 dx_2 \\ &= \frac{1}{4\pi\sigma_1\sigma_0} \int_{L(x_1,x_2)\geq\tau} \left[\exp\left(-\frac{X_1^2}{2\sigma_1^2} - \frac{X_2^2}{2\sigma_0^2}\right) + \exp\left(-\frac{X_1^2}{2\sigma_0^2} - \frac{X_2^2}{2\sigma_1^2}\right)\right] dx_1 dx_2\end{aligned} \tag{21}$$

and

$$\begin{aligned}P_F &= P(H_1|H_0) \\ &= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1, x_2|H_0) dx_1 dx_2 \\ &= \frac{1}{2\pi\sigma_0^2} \int_{L(x_1,x_2)\geq\tau} \exp\left(-\frac{X_1^2}{2\sigma_0^2} - \frac{X_2^2}{2\sigma_0^2}\right) dx_1 dx_2\end{aligned} \tag{22}$$

Let $L(x_1, x_2) \geq \tau$, we have

$$\exp\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2} x_1^2\right) + \exp\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2} x_2^2\right) \geq \frac{2\tau\sigma_1}{\sigma_0} \tag{23}$$

And here we need divide the problem into 2 cases to discuss.

- 1) If $\sigma_1 > \sigma_0$, then the upper bound region is

$$|x| \geq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln\left(\frac{2\tau\sigma_1}{\sigma_0} - 1\right)} = C_1 \tag{24}$$

and the lower bound region is

$$|x| \geq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln\left(\frac{\tau\sigma_1}{\sigma_0}\right)} = C_2 \tag{25}$$

The upper bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{\pi\sigma_1\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 2\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 4Q(\frac{C_1}{\sigma_1})Q(\frac{C_1}{\sigma_0})
\end{aligned} \tag{26}$$

The lower bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\geq \frac{1}{\pi\sigma_1\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 2\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 4Q(\frac{C_2}{\sigma_1})Q(\frac{C_2}{\sigma_0})
\end{aligned} \tag{27}$$

The upper bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= Q^2(\frac{C_1}{\sigma_0})
\end{aligned} \tag{28}$$

The lower bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\geq \frac{1}{2\pi\sigma_0^2}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= Q^2(\frac{C_2}{\sigma_0})
\end{aligned} \tag{29}$$

- 2) If $\sigma_1 < \sigma_0$, then the upper bound region is

$$|x| \leq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{2\tau\sigma_1}{\sigma_0}-1\right)} = C_1 \tag{30}$$

and the lower bound region is

$$|x| \leq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\tau\sigma_1}{\sigma_0}\right)} = C_2 \tag{31}$$

The upper bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{4\pi\sigma_1\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \left(1-2Q(\frac{C_1}{\sigma_1})\right)\left(1-2Q(\frac{C_1}{\sigma_0})\right)
\end{aligned} \tag{32}$$

The lower bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{4\pi\sigma_1\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \left(1-2Q(\frac{C_2}{\sigma_1})\right)\left(1-2Q(\frac{C_2}{\sigma_0})\right)
\end{aligned} \tag{33}$$

The upper bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \left(1-2Q(\frac{C_1}{\sigma_0})\right)^2
\end{aligned} \tag{34}$$

The lower bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \left(1-2Q(\frac{C_2}{\sigma_0})\right)^2
\end{aligned} \tag{35}$$

Exercise 2.2.19

Part 1 The LRT is

$$\begin{aligned}
L(R) &= \prod_{i=1}^{N}\frac{p_{r_i|H_1}(R_i|H_1)}{p_{r_i|H_0}(R_i|H_0)} \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N\prod_{i=1}^{N}\exp\left[\frac{(R_i-m_0)^2}{2\sigma_0^2}-\frac{(R_i-m_1)^2}{2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N\prod_{i=1}^{N}\exp\left[\frac{\sigma_1^2(R_i-m_0)^2-\sigma_0^2(R_i-m_1)^2}{2\sigma_0^2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N\prod_{i=1}^{N}\exp\left[\frac{(\sigma_1^2-\sigma_0^2)R_i^2-2R_i(\sigma_1^2m_0-\sigma_0^2m_1)+\sigma_1^2m_0^2-\sigma_0^2m_1^2}{2\sigma_0^2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N\exp\left[\frac{(\sigma_1^2-\sigma_0^2)l_\beta-2l_\alpha(\sigma_1^2m_0-\sigma_0^2m_1)+N\sigma_1^2m_0^2-N\sigma_0^2m_1^2}{2\sigma_0^2\sigma_1^2}\right]
\end{aligned} \tag{36}$$

Part 2 If $2m_0 = m_1$, $2\sigma_1 = \sigma_0$, we can derive the LRT is

$$\begin{aligned}
L(R) &= 2^N\mathsf{exp}\left(\frac{-3l_\beta+14l_\alpha m_0-15Nm_0^2}{8\sigma_1^2}\right)\underset{H_0}{\overset{H_1}{\gtrless}}\tau \\
\Longrightarrow l_\beta &\underset{H_1}{\overset{H_0}{\gtrless}} -\frac{8}{3}\sigma_1^2(\ln\tau-N\ln 2)+\frac{14}{3}l_\alpha m_0-5Nm_0^2.
\end{aligned} \tag{37}$$

Therefore, there is a line separate the l_α, l_β-plane to two parts. The slope and the intercept of the line are

$$\begin{cases} k = \frac{14}{3}m_0 \\ b = -\frac{8}{3}(\ln\tau-N\ln 2)\sigma_1^2-5Nm_0^2 \end{cases} \tag{38}$$

The region above the line is H_0 and the other is H_1.